colourful

Mathematics 12

MOZAIK

József Kosztolányi
István Kovács
Klára Pintér
János Urbán
István Vincze

Mathematics

textbook

12

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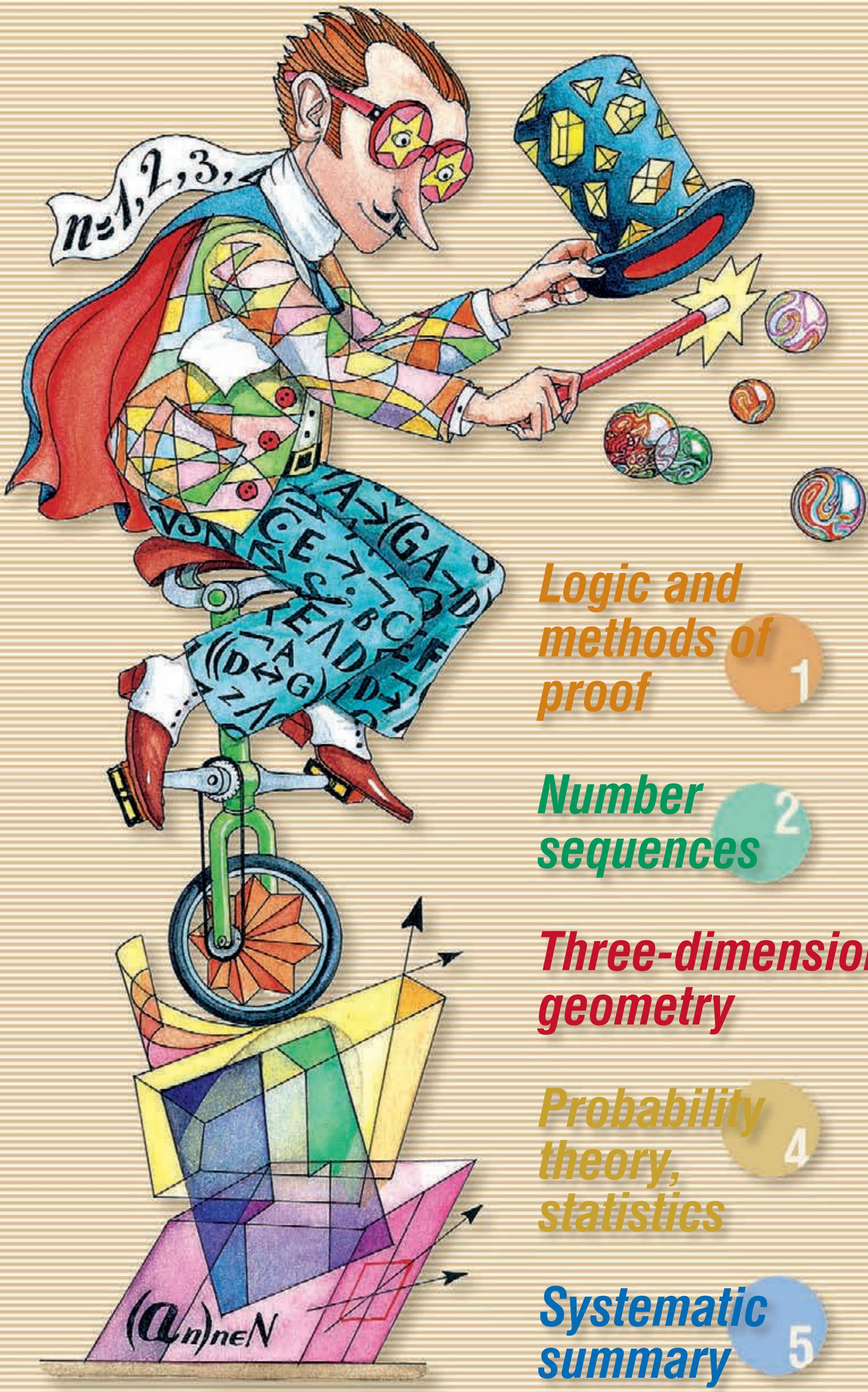
n=1,2,3,
(an)n∈N

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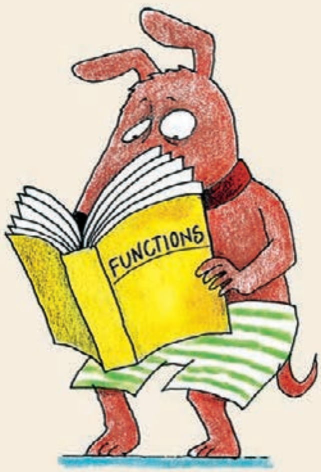


Guide to use the course book

The notations and highlights used in the book help with acquiring the courseware.

- The train of thought of the worked examples shows samples how to understand the methods and processes and how to solve the subsequent exercises.
- The most important definitions and theorems are denoted by colourful highlights.
- The parts of the courseware in small print and the worked examples noted in claret colour help with deeper understanding of the courseware. These pieces of knowledge are necessary for the higher level of graduation.
- Figures, the key points of the given lesson, review and explanatory parts along with interesting facts of the history of Mathematics can be found on the margin.

The difficulty level of the examples and of the appointed exercises is denoted by three different colours:

Yellow: drilling exercises with basic level difficulty; the solution and drilling of these exercises is essential for the progress.
Blue: exercises the difficulty of which corresponds to the intermediate level of graduation.
Claret: problems and exercises that help with preparing for the higher level of graduation.

These colour codes correspond to the notations used in the Colourful mathematics workbooks of Mozaik Kiadó. The workbook series contains more than 3000 exercises, which are suitable for drilling, working on in lessons and which help with preparing for the graduation.

◆ ◆ ◆

Mathematics, "ratio" and logic way of thinking are probably the most efficient tools of cognition of our world, which sometimes associate with unexplainable phenomena. These are inseparable from Homo sapiens and these make the everyday activities complete.

A few thoughts from those who have experienced all this:

"Our job is to hand over the torch of mathematical knowledge to the engineers, scientists, teachers and last but not least, mathematicians of the future. Are problems helpful in this task? Definitely they are. The major part of all sensible lives consists of solving problems, a large part of the job done by engineers and scientists is nothing but solving mathematical problems." (Paul R. Halmos)

"Every man ought to be inquisitive through every hour of his great adventure down to the day when he shall no longer cast a shadow in the sun. For if he dies without a question in his heart, what excuse is there for his continuance?" (Frank Moore Colby)

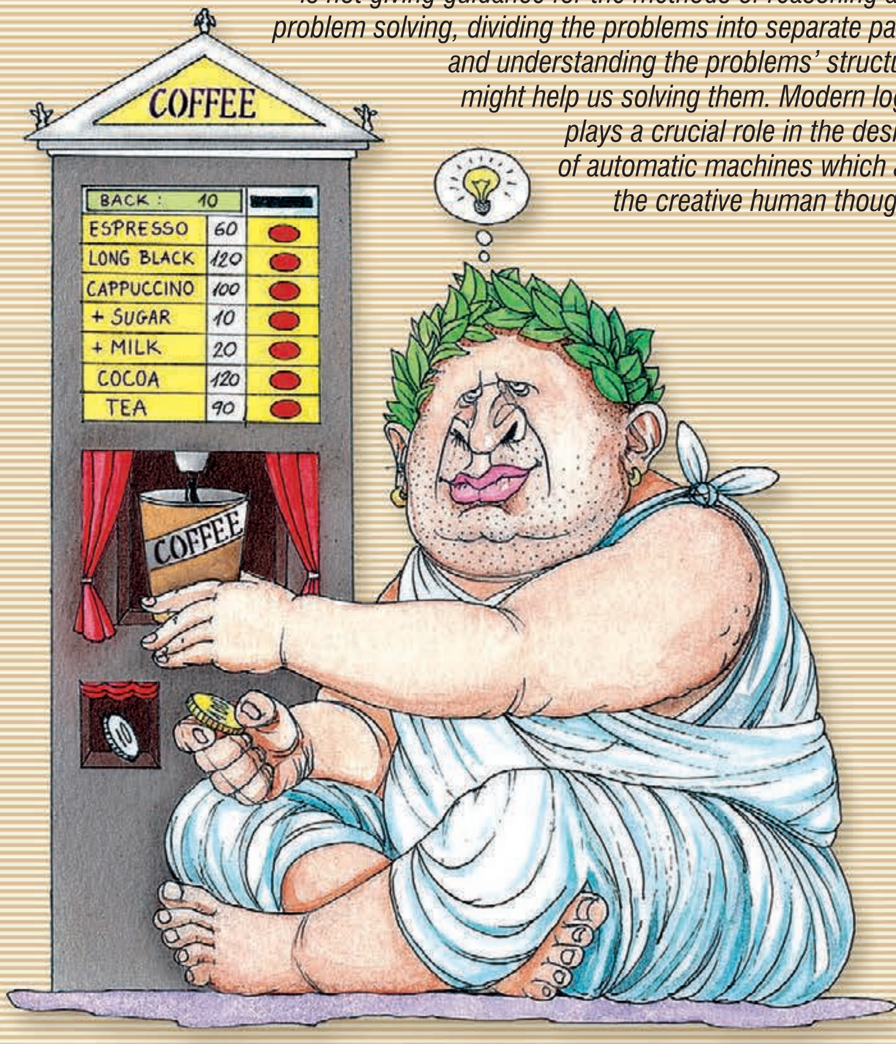
"I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." (Sir Isaac Newton)

"Maybe the result is not as important in mathematics as the way it takes to obtain that result." (Ervin Fried)

The Authors wish productive work and learning.

Logic and methods of proof

The word "logic" comes from the Greek term "logos", which means word, thought or truth. The Greek word "logike" means reasoning or conclusion. In Ancient science, philosophy gave rise to the questions whose study led to the development of logic. Logic is principally used in evaluating whether a conclusion is true or not, and in the critical analysis of the results of reasoning. Logic helps us to make precise, unambiguous definitions and statements, and as such, it is essential in scientific communication. Despite logic is not giving guidance for the methods of reasoning and problem solving, dividing the problems into separate parts and understanding the problems' structure might help us solving them. Modern logic plays a crucial role in the design of automatic machines which aid the creative human thought.



3. Logical operators: implication and equivalence

Mathematics frequently uses statements with on *"if… then…"* structure. For example, *"If a number is divisible by 6, then it is divisible by 3 as well."* Let us take a look at this type of statements from a logical point of view.

Example 1

Annie's cat purrs every morning if that day there will be rain. Today the cat purrs. Should Annie bring an umbrella?

Solution

The cat purrs when there will be rain. This does not tell anything about the situation when there is no rain, so if there will be no rain, the cat either purrs or not. Thus Annie's cat might purr even if no rainfall shall be expected, so Annie should not take for sure that she will need her umbrella.

Example 2

We come across two inhabitants of Luth Island. The taller one says: *"If I am a knight, then my companion is a knight as well."* Do we know what they are?

Solution

If the taller inhabitant is a knight, his statement is true, and since he is a knight, his companion has to be a knight as well. If the taller man is a knave, the condition of the statement (that he is a knight) is not true for him, thus no matter what his companion is, the statement will be true. But a knave cannot tell the truth.

Therefore the only possibility is that both inhabitants are knights.

From the statement *"If I am a knight then p"* it follows that *p* is true.

Example 3

On Luth Island, we meet two inhabitants: a man and a lady. The lady tells: *"If this man is a knight, then I am a knave."* Can we determine what they are?

Solution

If the man is a knight and the lady is a knave then her statement is true, however this is impossible since knaves cannot tell the truth.

If both of them are knights, the statement is false, which is impossible because knights cannot lie.

If the man is a knave, the first condition of the statement is not fulfilled and therefore the lady can be whatever, the statement will be true. Since the statement is true, the lady must be a knight.

Thus the lady is a knight and the man is a knave.

implication

if A then B =
= B if A =
= B, supposing A =
= A is a sufficient condition of B

DEFINITION: **The logical operation which equals „if *A* then *B*” is called *implication,* in which A is the *antecedent* and B is the *consequent.* The logical value of the implication if false exactly when its antecedent is true and its consequent is false, otherwise the implication is true.**

The implication from *A* to *B* is written as $A \to B$ and pronounced as “*A* implies *B*”, or “if *A*, then *B*”.

A	B	$A \to B$
t	*t*	*t*
t	*f*	*f*
f	*t*	*t*
f	*f*	*t*

Example 4

Is the following implication true?

If the moon is made of cheese, then this is a mathematics book.

Solution

Since the antecedent of the implication, A = *the moon is made of cheese,* is false, therefore, regardless of the truth value of the consequent, B = *this is a mathematics book,* the implication $A \to B$ is true.

Obviously this does not mean that $B \to A$ is also true, as this is a mathematics book, however the moon is not made of cheese.

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implication is non-commutative

We do not guarantee the safety of belongings not placed in the luggage room.

Does it follow from this that they guarantee the safety of those placed in the room?

Implication is a **non-commutative** operator, so $A \to B$ is not equivalent $B \to A$.

Implication is a **non-associative** operator, so $(A \to B) \to C$ is not equivalent to $A \to (B \to C)$.

Implication is non-associative

$A \to B$ = A is a sufficient condition of B

$B \to A$ = A is a necessary condition of B

Example 5

Describe the following statements with logical operators, if T = Timbertoes FC wins the cup and H = I'll eat my hat.

a) If Timbertoes FC wins the cup then I'll eat my hat.
b) Timbertoes FC will not win or I'll eat my hat.
c) If I don't eat my hat, Timbertoes FC will not win.
d) It is not true that Timbertoes FC wins and I don't eat my hat.

Solution

a) $T \to H$; *b)* $\neg T \vee H$; *c)* $\neg H \to \neg T$; *d)* $\neg(T \wedge \neg H)$.

Observe that all statements become false in only one case: when Timbertoes FC wins but I do not eat my hat.

The following is true in general:

$A \to B \equiv \neg A \vee B$

THEOREM: **For any statements A and B**

$$A \to B \equiv \neg A \vee B.$$

In mathematics we frequently use the term *"if and only if"*, e.g. *"A natural number is divisible by 9 if and only if the sum of its digits is divisible by 9."* On one hand, this means that if the number is divisible by 9, then the sum of its digits is also divisible by 9, and on the other hand, if the sum of the number's digits is divisible by 9 then the number itself is divisible by 9.

Example 6

On Luth Island, the traveller meets an inhabitant and asks him whether there is gold on the island. The inhabitant replies: *"If and only if I am a knight, then there is gold on the island."* What does the traveller learn from this?

Solution

In other words the inhabitant tells the following: *"If I am a knight then there is gold on the island, and if there is gold on the island then I am a knight."*

If the inhabitant is a knight then his statement is true, and since it is true that he is a knight, it will also be true that there is gold on the island.

From the statement *"I am a knight if and only if p"* it follows that p is true.

If the inhabitant is a knave, it is not true that he is a knight. In this case, the statement *"if I am a knight then there is gold on the island"* must be true.

If it was false that there is gold on the island, then the statement *"if there is gold on the island then I am a knight"* would be also true, thus the statement *"if and only if I am a knight, there is gold on the island"* made by the knave would have to be also true, which is impossible.

Therefore the inhabitant is a knight and there is gold on the island.

equivalence

Definition: **The logical operation which equals the logical relationship "A if and only if B" is called *equivalence*. Logical value of the equivalence is true exactly when A and B have the same logical value, otherwise the equivalence is false.**

The equivalence of A and B is written as $A \leftrightarrow B$ and pronounced as "A equals B", or „A if and only if B".

A if and only if B =
= A is equivalent to B =
= A is a sufficient and necessary condition of B

A	B	$A \leftrightarrow B$
t	t	t
t	f	f
f	t	f
f	f	t

Based upon the truth table, the **characteristics of the equivalence operation** can be easily verified for any statements A, B and C:

(1) **commutativity:** $A \leftrightarrow B \equiv B \leftrightarrow A$;

(2) **associativity:** $A \leftrightarrow (B \leftrightarrow C) \equiv (A \leftrightarrow B) \leftrightarrow C$;

(3) $A \leftrightarrow A \equiv t$; $A \leftrightarrow t = A$; $A \leftrightarrow f = \neg A$.

The statement *"A if and only if B"* means first that *if A, then B* and second, that *if B, then A*. This is expressed in the following:

Theorem: **For any A, B statements:**
$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

Example 7

Describe the logical structure of the following composite statements:

a) If –2 is greater than zero then –2 is not negative.

b) If –2 is less than zero then –2 is negative.

c) If and only if –2 is positive or equal to zero, then it is not negative.

d) –2 is not equal to zero if and only if it is positive or negative.

Solution

Introduce the following symbols:

p = –2 is greater than zero; q = –2 is less than zero;
r = –2 is negative; s = –2 is positive;
t = –2 is zero.

a) $p\to\neg q$; *b)* $q\to r$; *c)* $(s\lor t)\leftrightarrow\neg r$; *d)* $\neg t\leftrightarrow(r\lor s)$.

Notice that all statements are true!

Exercises

1. Formulate the following statements if A = the ice is at least 8 inches on the lake, B = I am going to skate on the lake.

a) There has to be at least 8 inches of ice on the lake for me going to skate on it.

b) If there is not at least 8 inches of ice on the lake then I am not going to skate on it.

c) I am going to skate on the lake if there is at least 8 inches of ice on it.

d) There is at least 8 inches of ice on the lake or else I am not going to skate on it.

2. Let A = today is Friday and B = tomorrow is Saturday. Formulate the following sentences using logical operators:

a) If today is Friday then tomorrow is Saturday.

b) If tomorrow is not Saturday then today is not Friday.

c) For tomorrow being Saturday it is necessary that today is Friday.

d) Tomorrow is Saturday, this is sufficient for today being Friday.

e) Tomorrow is not Saturday, this is sufficient for today not being Friday.

f) Tomorrow is Saturday if and only if today is Friday.

g) Today is Friday, which is sufficient and necessary for tomorrow being Saturday.

3. Let:

A = number n is divisible by 12. B = the last two digits of the number n are 36.
C = number n is prime. D = number n is even.
E = number n is divisible by 4. F = number n is divisible by 6.
G = the sum of n's digits is divisible by 3.

Find the statements belonging to the following expressions:

a) $B\to E$; *b)* $A\to\neg C$; *c)* $E\to(\neg C\land D)$;
d) $(D\land G)\leftrightarrow F$; *e)* $A\leftrightarrow(E\land G)$; *f)* $(\neg D\land G)\to\neg F$.

4. Write down the logical structure of the following complex statements:
   a) If a quadrilateral is a rectangle and its neighbouring sides have the same length, then it is a square.
   b) A triangle is right angled if and only if the sum of the square of two sides' lengths equals the square of the last side's length.
   c) If a number's square is greater than 4, then the number itself is either greater than 2 or less than –2.
   d) If the product of two integers is even then they cannot be both odd.

5. Five children made the following statements about a positive integer:
   Steve: It is divisible by 3.
   Johnny: It is divisible by 4.
   Will: It is divisible by 6.
   Kathy: It is divisible by 9.
   Betsy: It is divisible by 12.
   Who was mistaken if we know that there was exactly one false statement?

6. If Andy is a boy then Andy is younger than Susy. If Andy is 18 years old then Andy is a girl. If Andy is not 18 years old then Andy is at least as old as Susy. Can we decide from what we know whether Andy is a boy or a girl?

7. We ask a resident on Luth Island *"Is the statement 'There is gold on this island' equivalent with the statement 'You are a knight'?"* What does it mean if the resident replies with *yes*? What about *no*?

Puzzle

We see the following four cards: $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$.

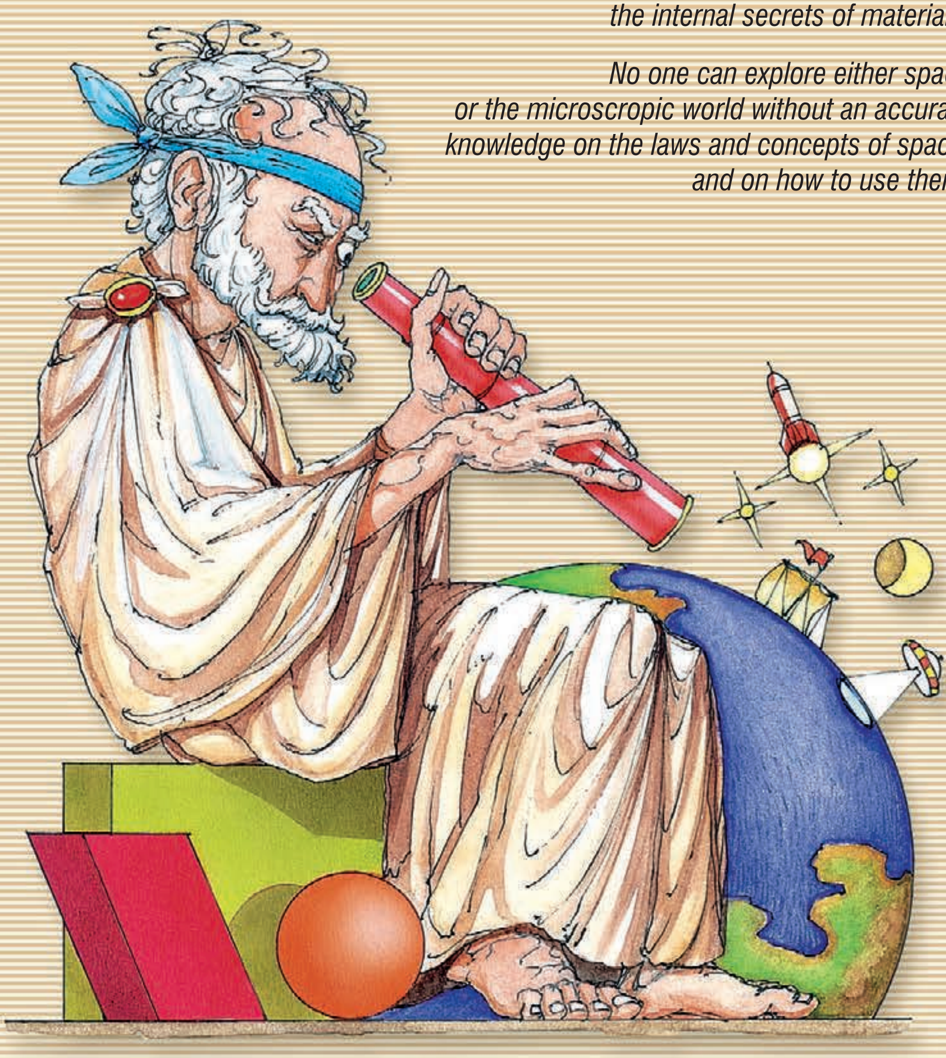
On each card, there is one number out of 1, 2, 3 and 4 on each side. At least how many cards do we have to flip so that we can decide whether the following statement is true or false: *"If there is 2 on one side of a card then there is 4 on the other side"*?

Three-dimensional geometry

Geometry is one of the most ancient branches of mathematics. Though scientific discoveries have expanded boundaries of research, we explore space exactly the same way as millennia before.

Aristotle found special only that when a ship sails against the horizon, its hull is the first to disappear and the mast is the last. In our age, spacecrafts orbit around Mars and ever-larger resolution microscopes discover the internal secrets of materials.

No one can explore either space or the microscropic world without an accurate knowledge on the laws and concepts of space, and on how to use them.



6. Concept of volume, the volume of the prism and the cylinder

Similarly to the area, which assigns a measure to planar objects, we might need to define some measure which describes their size. This can affect their value, aesthetic role or usage. One of the important information about a diamond or a pearl is its size. When creating a workplace, its size depends on the number of people that will use it.

The volume of solids can be defined similarly to the area of polygons. First we define the volume of polyhedra.

The *volume* of a polyhedron is a positive number describing the polyhedron, which satisfies the following:

(1) **The volume of the unit cube is 1.**

(2) **The volume of congruent polyhedra is equal.**

(3) **If a polyhedron is divided into sub-polyhedra, then the total volume of the parts equals the volume of the original polyhedron.**

It can be shown that for any polyhedron, one can assign a uniquely determined positive number satisfying the conditions, therefore every polyhedra has a uniquely determined volume.

It can also be shown that the above mapping is uniquely determined even for solids with curved surface. Thus cylindrical solids and cones also have volume. The proof of this statement is quite hard (it requires advanced mathematical tools), so for now we have to accept it without proof.

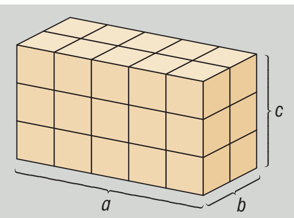
The volume of the cuboid

Determining the volume of a polyhedron can be viewed as a comparison to the volume of the unit cube. At a start, let us determine the volume of the cuboid.

volume of the cuboid

THEOREM: **The *volume of the cuboid* equals to the product of the lengths of the edges meeting in a vertex:** $V = a \cdot b \cdot c$.

Figure 40



Proof

The statement of the theorem is clearly true if the cuboid can be divided into unit cubes, that is, if its edges are all integers in length. *(Figure 40)*

If this is not the case, than we need more refined tools to complete the proof, but the statement is still valid.

The surface of a cuboid is equal to the sum of the areas of its bounding faces *(Figure 41)*. This sum, according to the net of the cuboid is

$$A = 2 \cdot (ab + ac + bc).$$

In the special case of the cube where the edges are equal in length: $a = b = c$, the volume is

$$\boxed{V = a^3.}$$

The size of the surface is

$$\boxed{A = 6 \cdot a^2.}$$

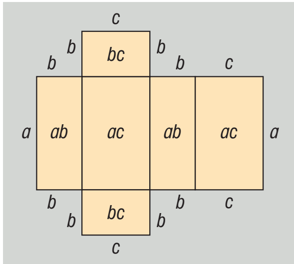


Figure 41

Example 1

We are to construct a cuboid using 30 congruent cubes. How should we proceed if we want to use as little paint to colour the resulting solid as possible?

Solution

According the condition, we have to try to minimize the surface of the resulting cuboid. Let us think of the cubes as unit cubes, then the volume of the cuboid is 30.

Examine in how many ways 30 can be expressed as the product of 3 positive integers:

$$30 = 1 \cdot 1 \cdot 30 = 1 \cdot 2 \cdot 15 = 1 \cdot 3 \cdot 10 = 1 \cdot 5 \cdot 6 = 2 \cdot 3 \cdot 5.$$

The possibilities are shown in Figure 42.

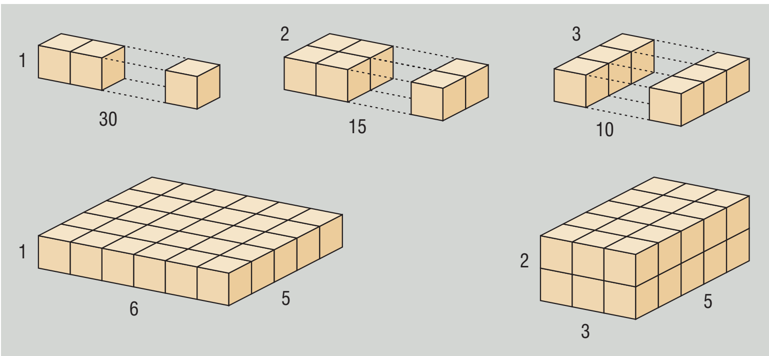


Figure 42

According to the formula $A = 2 \cdot (ab + ac + bc)$ of the surface, the surface areas of the constructions are as follows:

$A_1 = 2 \cdot (1 + 30 + 30) = 122;$ $\quad A_2 = 2 \cdot (2 + 15 + 30) = 94;$
$A_3 = 2 \cdot (3 + 10 + 30) = 86;$ $\quad A_4 = 2 \cdot (5 + 6 + 30) = 82;$
$A_5 = 2 \cdot (6 + 10 + 15) = 62.$

It can be concluded that the surface is minimal if the edges of the cuboid are $a = 2, b = 3, c = 5$.

Example 2

In a cuboid, the ratio of the edges meeting in one vertex is 4 : 5 : 20. The length of its diagonal is 42. Determine its volume and surface.

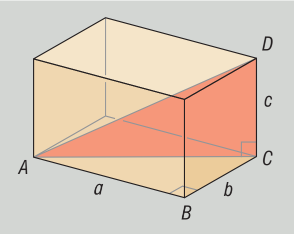


Figure 43

Solution

Let the length of the edges be denoted by a, b and c. ***(Figure 43)***

By using the Pythagorean theorem, for the diagonal AC of the base face we have

$$AC^2 = AB^2 + BC^2 = a^2 + b^2.$$

The space diagonal AD can be expressed using the right angled triangle ACD:

$$AD^2 = AC^2 + CD^2 = a^2 + b^2 + c^2.$$

This means that the square of length of the space diagonal equals the sum of the squares of the edges. Using the given ratio, let us denote the lengths of the edges by $4x$, $5x$ and $20x$. Substituting into the equation for the space diagonal gives

$$42^2 = (4x)^2 + (5x)^2 + (20x)^2,$$

$$1764 = 441x^2,$$

$$x^2 = 4.$$

The solution satisfying the conditions of the problem is $x = 2$, which means that the edges of the cuboid are $a = 8, b = 10, c = 40$.

Therefore the volume is

$$V = a \cdot b \cdot c = 3\,200.$$

The surface area is

$$A = 2 \cdot (ab + ac + bc) = 1600.$$

The volume of the prism

For a cuboid, if we think of the face with edges a and b as base, then its altitude is c. Then the volume of the cuboid can be viewed as the product of the area of its base, ab, and its altitude, c. This equality is true for every prism as well.

volume of a prism

THEOREM: **The *volume of a prism* is the product of the area of its base and the corresponding altitude:**
$$V = A_{\text{base}} \cdot m.$$

This statement holds for right prism, for oblique prism and for every cylindrical solids as well. The proof of this fact is not easy, the principles needed to verify are beyond the level of this book; although we will now present one of them as it will prove useful in the following.

At the start of the 17th century, Italian mathematician Cavalieri stated a proposition concerning the volumes of different types of solids. This has since been called Cavalieri's principle.

CAVALIERI'S PRINCIPLE: If, for two solids, there exists a plane such that the area of the intersection of one solid with any of the planes parallel to the first plane equals the area of the intersection of the same plane with the other solid, then the two solids have equal volume. *(Figure 44)*

As a student of Galilei, Italian mathematician FRANCESCO BONAVENTURA CAVALIERI (1598–1647) helped to formulate his master's results from physics in the language of mathematics. He imagined that solids consist of elements of one less dimension, which lead to the formulation of the principle named after him.

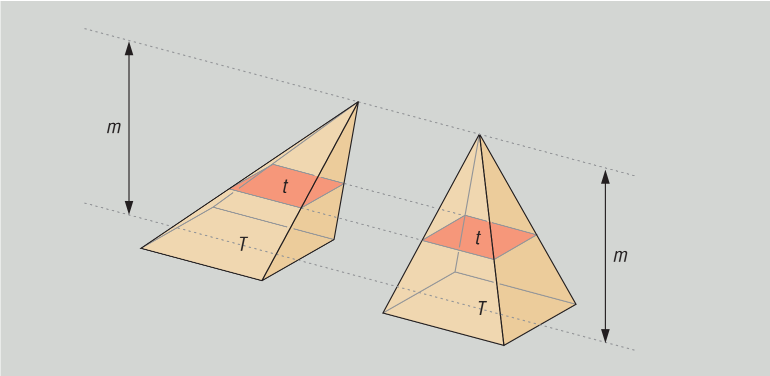


Figure 44

We shall note that we would need extra conditions to state the most general form of the principle, but this form is enough for determine the volume of the solid we are interested in.

Example 3

The volume of two prisms, one with a base of an equilateral triangle and the other with a base of a square are equal, as well as the length of the edges of their bases. Find the ratio of their altitudes.

Solution

Let us denote the length of the edges of the bases by a, the altitude of the triangular prism by m_1 and the altitude of the square prism by m_2.

The volumes of the two solids are, respectively,

$$V_1 = t_1 \cdot m_1 = \frac{a^2 \cdot \sqrt{3}}{4} \cdot m_1, \qquad V_2 = t_2 \cdot m_2 = a^2 \cdot m_2.$$

Since the volumes are equal, it follows that

$$\frac{a^2 \cdot \sqrt{3}}{4} \cdot m_1 = a^2 \cdot m_2.$$

From this, we can express the ratio of the altitudes:

$$\frac{m_1}{m_2} = \frac{a^2}{\frac{a^2 \cdot \sqrt{3}}{4}} = \frac{4}{\sqrt{3}} = \frac{4 \cdot \sqrt{3}}{3}.$$

The volume of the cylinder

The volume of cylinders can be determined with the help of the prism: it holds that any cylinder can be approximated with arbitrary precision using an appropriately chosen prism. Thus the volume of a cylinder with radius r and altitude m is ***(Figure 45)***:

$$V = A_{\text{base}} \cdot m, \quad \text{that is,}$$

$$\boxed{V = r^2 \cdot \pi \cdot m.}$$

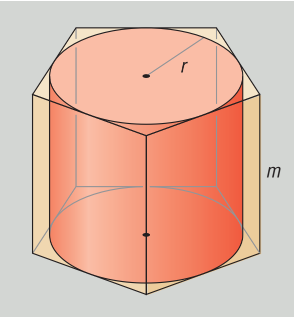


Figure 45

volume and surface of the cylinder

When determining the surface area, observe that the lateral surface is a rectangle with sides equal to the circumference of the base disc and the altitude in length ***(Figure 46)***; the latter of which is also the generatrix a of the right circular cylinder:

$$A = 2 \cdot A_{\text{base circle}} + A_{\text{generatrix}} = 2 \cdot r^2 \cdot \pi + 2 \cdot r \cdot \pi \cdot a, \quad \text{from which}$$

$$\boxed{A = 2 \cdot r \cdot \pi \cdot (r + a).}$$

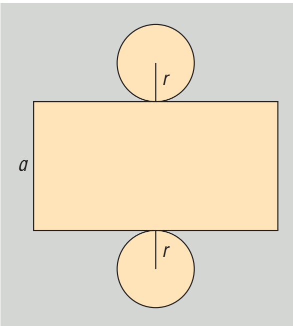


Figure 46

Example 4

The radius of a cylinder, with altitude being 2 m, is 1 m. The cylinder is filled with water up to the $\frac{1}{4}$ of the diagonal in a lying position. How high will the water be if we erect the cylinder?

Solution

The volume of the water in the lying cylinder is equal to the volume of a prism with a segment-shaped base and an altitude of 2 m. ***(Figure 47)***

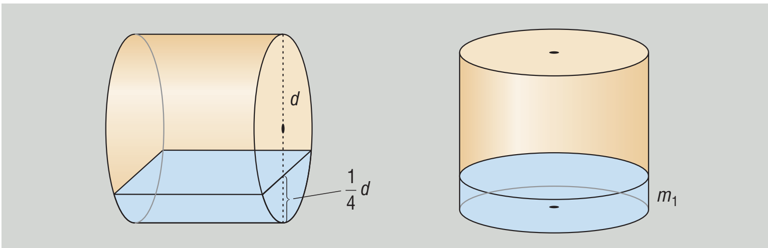
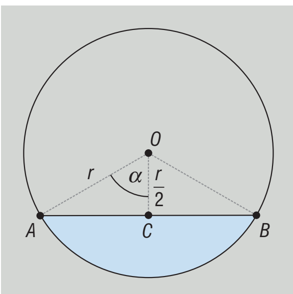


Figure 47

The area of the segment, and thus the area of the base, is the difference of the area of a sector and a triangle ***(Figure 48)***. The central angle of the sector can be found using triangle OAC: $\cos\alpha = \frac{\frac{r}{2}}{r} = \frac{1}{2}$, therefore $\alpha = 60^\circ$, and the central angle is $2\alpha = 120^\circ$. Thus the area of the sector is one third of the area of a full disc. The area of the triangle OAB equals the area of an equilateral triangle with sides of length r, therefore

$$A_{\text{segment}} = \frac{r^2 \cdot \pi}{3} - \frac{r^2 \cdot \sqrt{3}}{4} \approx 0.61 \text{ m}^2.$$

Figure 48



This means that the volume of the water in the cylinder is

$$V = A_{\text{segment}} \cdot m \approx 0.61 \cdot 2 = 1.22\ \text{m}^3.$$

In a standing position the base will be a disc with radius r. Let the height of the water in this case be m_1. Using the previously computed volume,

$$V = r^2 \cdot \pi \cdot m_1,$$

$$m_1 = \frac{V}{r^2 \cdot \pi} \approx \frac{1.22}{\pi} \approx 0.39\ \text{m}.$$

The height of the water will be 0.39 m.

Exercises

1. We would like to build a cuboid using 36 unit cubes. How many different ways are there? Which ones have the smallest and the largest surface area?

2. The ratio of the edges of a cuboid is $3:4:5$, the length of its space diagonal is 20. Determine the lengths of its edges. What is the volume and surface area of the cuboid? Determine the angle between the space diagonal and the shortest edge, and the angle between the space diagonal and the longest edge.

3. The ratio of the edges meeting in one vertex of a cuboid is $2:3:4$, its volume is $192\ \text{cm}^3$. Determine the lengths of its edges and its surface area.

4. The ratio of the edges of a cuboid with surface area $1300\ \text{cm}^2$ is $2:3:4$. Determine the lengths of its edges and its volume.

5. The edge of the base of a right prism is 10 cm long, its altitude is 20 cm. Find its volume and surface area if its base is
   a) an equilateral triangle;
   b) a regular pentagon;
   c) a regular hexagon;
   d) a regular 10-gon.

6. Determine the volume and surface area of the rotational cylinder with the following data given:
   a) $r = 5$ cm, $m = 10$ cm;
   b) $r = 10$ cm, $A_{\text{lateral surface}} = 2\,000\ \text{cm}^2$;
   c) $m = 10$ cm, $A_{\text{lateral surface}} = 1500\ \text{cm}^2$.

7. A cylinder is cut out of a cube with sides of length 10 cm. At least how many percent is the waste?

8. Take the inscribed and circumscribed cube of a sphere. The difference in the edge lengths of the cubes is 10 cm. What are the volumes and surface areas of the two cubes?

9. A rectangle with sides $a = 5$ cm and $b = 8$ cm is first rotated around its shorter, then around its longer edge. Find the volumes and surface areas of the resulting solids.

10. Place a plane on one of the edges of the base of a cube with edge length being 10 cm such that the angle between the plane and the base of the cube is 30º. What are the volumes of the two solids the cube is cut into by the plane? Determine the surface areas of the two solids.

11. The sum of the lengths of the three edges meeting in a vertex of a cuboid is 14, while the sum of their squares is 84. The length of one edge is the geometric mean of the other two. Determine the volume and surface area of the cuboid.

Thematic summary

The main goal of this thematic summary is to help preparation for the maturity exam at the end of Year 12. The four lessons summarize most of the topics from the last four years with some added extensions and generalizations as well.

Each lesson differs a little. In the lesson on algebra and number theory there are lots of examples as the studied definitions and theorems are best reviewed this way. In the lessons on functions and geometry our aim was to collect and sort the studied concepts, theorems, processes. In these lessons the examples mainly act as illustration.

In the thematic summary small letter or asterisk next to the number of an example or problem marks topics that go further than the main body of subject. These topics are not part of the intermediate level maturity exam but they are needed on the advanced level.

4. Operations with rational expressions

Expressions with integers (polynomials)

notable products

NOTABLE PRODUCTS:

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a+b)\cdot(a-b) = a^2 - b^2.$$

factorization

METHODS OF FACTORIZATION
- **factoring out;**
- **factoring out using grouping;**
- **using notable identities;**
- **using the factored form of a quadratic equation.**

Example 1

Expand the parentheses and group the terms:

a) $(2a-3)\cdot(4a+1)-(3a-5)$;

b) $(4b+3)\cdot(4b-3)-(3b-2)\cdot(3b+2)$.

Solution (a)

After expanding the parentheses and grouping the terms:

$$(2a-3)\cdot(4a+1)-(3a-5) = 8a^2+2a-12a-3-3a+5 =$$
$$= 8a^2-13a+2.$$

Solution (b)

Apply the identity for the product of sum and difference:

$$(4b+3)\cdot(4b-3)-(3b-2)\cdot(3b+2) =$$
$$= 16b^2-9-9b^2+4 = 7b^2-5.$$

Example 2

Compute the following and order the terms of the resulting polynomials in a decreasing order of powers of x:

a) $(3x+2)^2-(2x-3)^2$; *b)* $(2x-1)^3-(x+4)^3$; *c)* $(2x+y-3)^2$.

Solution

a) $(3x+2)^2-(2x-3)^2 = 9x^2+12x+4-(4x^2-12x+9) =$
$= 9x^2+12x+4-4x^2+12x-9 = 5x^2+24x-5;$

b) $(2x-1)^3-(x+4)^3 = 8x^3-12x^2+6x-1-(x^3+12x^2+48x+64) =$
$= 8x^3-12x^2+6x-1-x^3-12x^2-48x-64 = 7x^3-24x^2-42x-65;$

c) $(2x+y-3)^2 = 4x^2+y^2+9+4xy-12x-6y =$
$= 4x^2+4xy-12x+y^2-6y+9.$

Example 3

Which two-term expression's square is it:

a) $25x^2+10x+1;$ b) $x^6-14x^3+49;$ c) $x^4-8x^2y+16y^2$?

Solution

a) $25x^2+10x+1 = (5x+1)^2 = (-5x-1)^2;$

b) $x^6-14x^3+49 = (x^3-7)^2 = (7-x^3)^2;$

c) $x^4-8x^2y+16y^2 = (x^2-4y)^2 = (4y-x^2)^2.$

Example 4

Factorize the following expressions:

a) $6x^2-4x^3+8x^2;$ b) $2x^2+4x+xy+2y;$ c) $9x^2-25y^2;$
d) $3x^2-12x+12;$ e) $x^2+2x-15.$

Solution (a)

Factor out $2x^2$ from every term:

$$6x^4-4x^3+8x^2 = 2x^2\cdot(3x^2-2x+4).$$

The discriminant of the quadratic equation $3x^2-2x+4=0$ is $D=4-48=-44,$ thus the equation does not have a solution, therefore the quadratic polynomial $3x^2-2x+4$ cannot be factorized.

Solution (b)

Factor out $2x$ from the first two terms and y from the last two, and after a second factoring:

$$2x^2+4x+xy+2y = 2x\cdot(x+2)+y\cdot(x+2) =$$
$$= (x+2)\cdot(2x+y).$$

Solution (c)

Notice the notable product:

$$9x^2-25y^2 = (3x+5y)\cdot(3x-5y).$$

Solution (d)

After factoring 3 out the parentheses contain the square of a two-term expression:

$$3x^2-12x+12x+12 = 3\cdot(x^2-4x+4) = 3\cdot(x-2)^2.$$

Solution (e)

Method I

Complete the square and apply some notable product identities:

$$x^2+2x-15=x^2+2x+1-1-15=(x+1)^2-16=$$
$$=(x+1)^2-4^2=(x+1+4)\cdot(x+1-4)=(x+5)\cdot(x-3).$$

Method II

Use the factored form of a quadratic equation.

The factored form of the quadratic equation $ax^2+bx+c=0$, if $D\geq 0$:

$a\cdot(x-x_1)\cdot(x-x_2)=0$,

where x_1 and x_2 are the roots of the quadratic equation.

Solve the equation $x^2+2x-15=0$:

$$x_{1,2}=\frac{-2\pm\sqrt{4+60}}{2}=\begin{cases}x_1=3,\\ x_2=-5.\end{cases}$$

According to the factored form,

$$x^2+2x-15=(x-3)\cdot(x+5).$$

Algebraic fractions

Example 5

Simplify the following fractions:

a) $\dfrac{a^2+4a+4}{3a+6}$;   *b)* $\dfrac{2b^3-12b^2+18b}{b^3-9b}$.

Solution (a)

The first step is to find the domain: $3a+6\neq 0$, that is, $a\neq -2$.

Factorize both the numerator and the denominator, and simplify with the common factors:

$$\frac{a^2+4a+4}{3a+6}=\frac{(a+2)^2}{3\cdot(a+2)}=\frac{a+2}{3}.$$

Solution (b)

The denominator cannot be zero: $b^3-9b\neq 0$.

Using factorization:

$$b^3-9b=b\cdot(b^2-9)=b\cdot(b+3)\cdot(b-3).$$

Thus $b\neq 0,\ b\neq -3,\ b\neq 3.$

Factoring out helps us factorize the numerator as well:

$$\frac{2b^3-12b^2\cdot 318b}{b^3-9b}=\frac{2b\cdot(b^2-6b+9)}{b\cdot(b-3)\cdot(b+3)}=$$
$$=\frac{2b\cdot(b-3)^2}{b\cdot(b-3)\cdot(b+3)}=\frac{2\cdot(b-3)}{b+3}.$$

Example 6

Simplify the following expression as much as possible:

$$\frac{2a+1}{a-3}-\frac{a-1}{a+3}-\frac{11a+9}{a^2-9}.$$

Solution

The fractions are valid if their denominators are not equal to zero: $a \neq 3$ and $a \neq -3$.

Since $a^2-9=(a-3)\cdot(a+3)$, this product can be the common denominator:

$$\frac{2a+1}{a-3}-\frac{a-1}{a+3}-\frac{11a+9}{a^2-9}=$$

$$=\frac{(2a+1)\cdot(a+3)}{(a-3)\cdot(a+3)}-\frac{(a-1)\cdot(a-3)}{(a-3)\cdot(a+3)}-\frac{11a+9}{(a-3)\cdot(a+3)}=$$

$$=\frac{2a^2+6a+a+3-a^2+3a-3-11a-9}{(a-3)\cdot(a+3)}=\frac{a^2-9}{(a-3)\cdot(a+3)}=1.$$

Example 7

Compute the expression $\frac{3b-1}{b^2-b}-\frac{4}{b+1}+\frac{2b-6}{b^2-1}$ for $b=\frac{1}{23}$.

Solution

Factorize the denominators:

$$b^2-b=b\cdot(b-1);$$

$$b^2-1=(b-1)\cdot(b+1).$$

It can be seen using the factored forms that the fractions are valid if $b \neq 0; b \neq 1; b \neq -1$. The value b given can be substituted, but first simplify the expression.

The common denominator is $b\cdot(b-1)\cdot(b+1)$.

$$\frac{3b-1}{b^2-b}-\frac{4}{b+1}+\frac{2b-6}{b^2-1}=$$

$$=\frac{(3b-1)\cdot(b+1)}{b\cdot(b-1)\cdot(b+1)}-\frac{4\cdot b\cdot(b-1)}{b\cdot(b-1)\cdot(b+1)}+\frac{(2b-6)}{b\cdot(b-1)\cdot(b+1)}=$$

$$=\frac{3b^2+3b-b-1-4b^2+4b+2b^2-6b}{b\cdot(b-1)\cdot(b+1)}=$$

$$=\frac{b^2-1}{b\cdot(b-1)\cdot(b+1)}=\frac{1}{b}.$$

If $b=\frac{1}{23}$, then substitution gives the value 23.

Exercises

1. Factorize the following expressions:

   a) $8a^2 - 6a$; *b)* $25b^4 - b^2$; *c)* $28c^2 + 84c + 63$.

2. Let a, b, c, d be four consecutive natural numbers in an increasing order. Prove that $a + b^2 + c^3$ is divisible by d^2.

3. Compute the following expressions for $x = 2004$:

   a) $\dfrac{2x^2 - 16x + 32}{4x - 16}$; *b)* $\dfrac{x^2 + 6x + 5}{x^2 + 10x + 25} + \dfrac{x + 9}{x + 5}$.

4. Simplify the following expressions:

   a) $\dfrac{2x - 1}{2x - 6} + \dfrac{x - 1}{x + 3} - \dfrac{2x^2 + 1}{x^2 - 9}$; *b)* $\dfrac{3}{b^2 + b} - \dfrac{5}{b^2 - b} + \dfrac{8}{b^3 - b}$;

   c) $\left(\dfrac{8}{x^2 - 4} - \dfrac{2}{3x - 6} + \dfrac{2}{x + 2}\right) \cdot \dfrac{x - 2}{x - 4}$.